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On the "Mean Field" Interpretation of Burgers' Equation[1]

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Fruitful analogies, partially first established by C. M. Newman,[1] between the variables, functions, and equations which describe the equilibrium properties of classical ferro- and antiferromagnets in the Mean Field Approximation (MFA) and those which describe the space-time evolution of compressible Burgers' liquids are developed here for one-dimensional systems. It is shown that the natural analogies are: magnetic field and position coordinate; ferro-/antiferromagnetic coupling constants and negative/positive times; free energy per spin and velocity potential; magnetization and velocity field; magnetic susceptibility and mass density. An unexpected consequence of these analogies is a derivation of the Morette–Van Hove relation. Another novelty is that they necessitate the investigation of weak solutions of Burgers' equation for negative times, corresponding to the Curie–Weiss transition in ferromagnets. This is achieved by solving the "final-value" problem of the homogenous Hamilton–Jacobi equation. Unification of the final- and initial-value problems results in an extended Hopf–Lax variational principle. It is shown that its applicability implies that the velocity potentials at time zero be Lipschitz continuous for the velocity field to be bounded. This is a rather mild condition for the class of physically interesting and functionally compatible velocity potentials, compatible in the sense of satisfying the Morette–Van Hove relation.

KEY WORDS: Mean Field Approximation; Hamilton–Jacobi equation; Morette–Van Hove relation; Hopf–Lax formula.

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1. INTRODUCTION

In an earlier paper,[2] the class of those solutions of the continuity and of the homogeneous Bernoulli or Burgers equation of an inviscid one-dimensional liquid which satisfy the Hopf–Lax variational principle and the Morette–Van Hove relation was introduced and illustrated through several examples and counter-examples. For such a system, this relation stipulates that the mass density $\rho(x, t)$ is proportional to, and has the same sign as, the second x-derivative of the velocity potential $S(x, t)$. It is readily verified that if $S(x, t)$ satisfies the homogeneous Bernoulli or Hamilton–Jacobi equation (5) then the corresponding $\rho(x, t)$ satisfies the continuity equation (8). Since $\rho(x, t) \geqslant 0$ and is piecewise continuous, the velocity field which satisfies Burgers' equation (2) is a non-decreasing function of x and consequently there are no shocks. In the language of fluid dynamics this would mean the occurrence of rarefaction waves only, a rather poor content of this class of solutions! The point is, however, that since the smooth solutions solve Cauchy (i.e., initial-value) problems, they are valid for $t \geqslant 0$. It is then natural to ask about $t < 0$.

The following examples illustrate the issue. The first is taken from[2] (Section 3, Fig. 3). It concerns the space-time evolution of a normalized density, $\rho(x, 0) = 1/2$ for $|x| \leqslant 1$ and $\rho(x, 0) = 0$ otherwise, and of the compatible velocity field $u(x, 0) = x$ for $|x| \leqslant 1$ and $u(x, 0) = x/|x|$ for $|x| > 1$. The solution is $u(x, t) = x/(1+t)$ and $\rho(x, t) = 1/2(1+t)$ for $|x| \leqslant 1+t$ and $\rho(x, t) = 0$, $u(x, t) = x/|x|$ for $|x| > 1+t$. It is clear that a singularity, or shock, occurs at the origin at $t = -1$. The second example is also taken from ref. 2 (Section 3, Fig. 6) with the parameter $\nu = 1$. Here the compatible initial conditions are $\rho(x, 0) = 1/2\cosh^2(x)$, $u(x, 0) = \tanh(x)$, and $S(x, 0) = \ln(\cosh(x))$. The implicit solution of Burgers' equation (2) which is obtained from the Hopf–Lax formula (10) is $u(x, t) = \tanh(x - tu(x, t))$, for $t \geqslant 0$. There is a striking analogy between this implicit equation and that of the magnetization m of an Ising ferromagnet treated in the Mean Field Approximation (MFA). If H and J are respectively the external magnetic field and the ferromagnetic coupling constant expressed in units of $k_B T$, the Boltzmann constant times the temperature, we have indeed $m(H, J) = \tanh(H + Jm(H, J))$. Knowing that there is a phase transition at $H = 0$ and at the critical value $J_c = 1$, to which would correspond a critical time $t_c = -1$, it is quite obvious that investigating the domain $t < 0$ will reveal the occurrence of interesting weak-type solutions of Burgers' equation. Conversely, the implicit solutions of a similar approximation for an antiferromagnet map onto the smooth solutions of $u(x, t)$ for $t > 0$.

In Section 2 and on the basis of the original work of C. M. Newman[1] several analogies are established between the variables, functions and equations which describe the MFA of a classical ferromagnet and those of a Burgers liquid. Besides magnetization and velocity field we will consider free energy and velocity potential but also magnetic susceptibility and mass density, which is a new analogy to the best of our knowledge. These analogies will lead to an alternative proof of the Morette–Van Hove relation.

In Section 3 a modified version of the Hopf–Lax formula is proposed with the aim of solving the final-value problem. This version is compared with the variational principle which governs the MFA of a classical ferromagnet. We then establish an extended Hopf–Lax formula which, given an admissible velocity potential at $t=0$ enables us to investigate solutions, weak or otherwise, for $-\infty < t < \infty$. This generalization necessitates the restriction of the admissible velocity potentials at $t=0$ to those which are Lipschitz continuous, according to a theorem quoted by L. C. Evans in ref. 3 (Theorem 7, p. 132) and for the velocity field to be bounded. Fortunately, there is a large class of compatible and physically interesting velocity potentials which satisfy this requirement.

Section 4 is dedicated to applications. First, a two-parameter family of admissible initial conditions is proposed, then a representative example is treated numerically and, to a certain extent, analytically. We conclude with an exhaustive description of the evolution of our model. At $t=-\infty$ there is a complete collapse of the liquid in a Dirac distribution. As time increases, part of the liquid "evaporates" and occupies density tails surrounding the Dirac peak the amplitude of which diminishes and disappears at $t=t_c$ ($=-1$ in our model). At t_c the density profile exhibits an algebraic singularity which disappears for $t > t_c$ and the density profile becomes smooth and expands steadily with an amplitude decreasing hyperbolically with time. This behavior is illustrated with several figures obtained by using numerical tools to treat the equations that define the evolution.

In order to make this paper self-contained, an appendix on the derivation of C. M. Newman's relation is added.

2. MEAN FIELD THEORY AND BURGERS' EQUATION

In 1986, C. M. Newman[1] published interesting analogies between the variables and the equations which describe the MFA of classical ferromagnets and Burgers' theory of viscous liquids. For convenience, Newman's relation is rederived in Appendix A. In the thermodynamic limit and with the convention that subscripts, except zero, designate partial derivatives, Newman's PDE is equivalent to

$$m_J - m m_H = 0. \tag{1}$$

The analog of (1) is Burgers' equation in the inviscid limit, namely

$$u_t + u u_x = 0. \tag{2}$$

Using $\leftrightarrow$ to represent analogy, we follow Newman in setting $u \leftrightarrow m$, but differ from him in setting $t \leftrightarrow -J$ and $x \leftrightarrow H$ because of the way the implicit solutions for $u(x, t)$ and $m(H, J)$ in Section 1 depend on their variables. Thus

$$u(x, -t) \leftrightarrow m(H, J). \tag{3}$$

There is another analogy between the free energy per spin $F(H, J)$ and the velocity potential $S(x, t)$. Clearly, since $m = -F_H$ and $u = S_x$, $F(H, J)$ satisfies the PDE

$$F_J + \tfrac{1}{2} F_H^2 = 0, \tag{4}$$

whereas $S(x, t)$ satisfies the homogeneous Bernoulli or Hamilton–Jacobi equation

$$S_t + \tfrac{1}{2} S_x^2 = 0. \tag{5}$$

As $x \leftrightarrow H$, $t \leftrightarrow -J$, $u = S_x$, and $m = -F_H$ we conclude that

$$-S(x, -t) \leftrightarrow F(H, J), \tag{6}$$

a relation which will be useful in the next section.

One more analogy, which is new to our knowledge and is perhaps the most interesting, is between the magnetic susceptibility $\chi = m_H$ and the mass density $\rho(x, t)$ of the liquid. Taking the H-derivative of (1) results in the PDE

$$\chi_J - (\chi m)_H = 0, \tag{7}$$

and from the continuity equation of the liquid

$$\rho_t + (\rho u)_x = 0, \tag{8}$$

it follows that

$$\rho(x, -t) \leftrightarrow \chi(H, J). \tag{9}$$

Since $\chi = m_H \leftrightarrow u_x$, $\rho = u_x = S_{xx}$ up to a proportionality constant. However this is precisely the Morette–Van Hove relation discussed in ref. 2! This means that, following the above analogies, we have found an alternative proof of this relation for the one-dimensional perfect liquid. For the general case see refs. 4 and 5.

3. HOPF–LAX AND THERMODYNAMIC VARIATIONAL PRINCIPLES

Consider first the initial-value problem of solving Hamilton–Jacobi's equation (5) with compatible and admissible (to be qualified below) initial conditions $S_0(x) = S(x, 0)$. Moreover let $(x-y)^2/2t$ be Hamilton's principal function for a characteristic, a straight line emanating from y at time zero and reaching x at time t. The Hopf–Lax variational principle, also called the Hopf–Lax formula, tells us in this case that

$$S(x, t) = \inf_{y \in \mathbb{R}} \left(S_0(y) + \frac{(x-y)^2}{2t} \right), \qquad 0 < t < \infty. \tag{10}$$

Let $S_{0y}(y) = u(y, 0) = u_0(y)$. The solution for y is given by the unique root of the equation

$$x = y + tu_0(y), \qquad 0 \leqslant t < \infty. \tag{11}$$

Since $u(x, t) = S_x(x, t) = \frac{x-y}{t}$ where y is the solution of (11), it follows that $u(x, t) = u_0(y(x, t))$ or

$$u(x, t) = u_0(x - tu(x, t)), \tag{12}$$

and, upon inversion,

$$x(u, t) = tu + u_0^{-1}(u), \tag{13}$$

where $u_0^{-1}(u)$ is the inverse function of $u = u_0(y)$. Lastly, and since $y = x - tu$, the Hopf–Lax formula (10) can also be written in the form

$$S(x, t) = \inf_{u \in D(u)} \left(S_0(x - tu) + \tfrac{1}{2} tu^2\right), \tag{14}$$

where $D(u)$ is the domain of u specified below.

In order to solve the final-value problem in (5), we set $t = -\bar{t}$ and since it is convenient to keep the operation $\inf_{y \in \mathbb{R}}$ in the variational principle, we change the signs in (10). Thus,

$$-S(x, -\bar{t}) = \inf_{y \in \mathbb{R}} \left(-S_0(y) + \frac{(x-y)^2}{2\bar{t}} \right). \tag{15}$$

However, (15) implies that $-S_0(x)$ is also an admissible velocity potential at time zero. Consequently $+S_0(x)$ and $-S_0(x)$ have to be admissible. Remarkably enough, L. C. Evans[3] (Theorem 7, uniqueness of weak solutions, p. 132) indicates that both these functions are admissible provided that they are Lipschitz continuous which means that $u(x, t)$ is bounded, i.e., that $D(u)$ is of compact support. This author also gives an explicit solution of (10) and (15) with $\pm|x|$ as initial conditions[3] (pp. 135–136). Fortunately there is a large class of compatible and physically meaningful initial conditions which satisfy this requirement. In particular it suffices that the density $\rho(x, 0) \leqslant 1/x^2$ at infinity. In Section 4 the case where $2\rho(x, 0) = 1/\cosh^2(x)$ will be chosen.

By analogy with (11)–(14),

$$x = y - \bar{t}u_0(y), \qquad 0 \leqslant \bar{t} < \infty, \tag{16}$$

$$u(x, -\bar{t}) = u_0(x + \bar{t}u(x, -\bar{t})), \tag{17}$$

$$x(u, -\bar{t}) = -\bar{t}u + u_0^{-1}(u), \tag{18}$$

and

$$-S(x, -\bar{t}) = \inf_{u \in D(u)} \left(-S_0(x + \bar{t}u) + \tfrac{1}{2}\bar{t}u^2\right). \tag{19}$$

At this point we wish to invoke the Hamilton–Jacobi analog equation (4) for the free energy $F(H, J)$. If $-F(H, 0) \equiv -F_0(H)$ is the entropy of a free spin in the magnetic field H and $-F_{0H}(H) \equiv m_0(H)$ its magnetization, then equation (4) implies that the Hopf–Lax formula can be used to solve the Cauchy problem of (4). Thus

$$F(H, J) = \inf_{K \in \mathbb{R}} \left(F_0(K) + \frac{1}{2} \frac{(H-K)^2}{J} \right), \tag{20}$$

where K is a variable magnetic field which, at the infimum of (20), turns out to be the Weiss internal field. If $J < J_c$, then K is the unique solution of

$$H = K - Jm_0(K). \tag{21}$$

Since $m(H, J) = -F_H(H, J) = \frac{K-H}{J} = m_0(K)$ according to (20) and (21), we have indeed

$$K(H, J) = H + Jm(H, J), \tag{22}$$

$$m(H, J) = m_0(H + Jm(H, J)), \tag{23}$$

$$H(m, J) = -Jm + m_0^{-1}(m). \tag{24}$$

The analog of (19) can also be found by introducing (22) in (20). The result is

$$F(H, J) = \inf_{m \in D(m)} \left(F_0(H + Jm) + \tfrac{1}{2} Jm^2\right), \tag{25}$$

which is the well known thermodynamic variational principle of the mean field theory for the order parameter m. It is now clear that a critical time $t = -\bar{t}_c$ corresponds to the critical J_c at which there is a phase transition.

It remains to combine the solutions of the initial- and of the final-value problems. For all times except 0

$$\frac{t}{|t|} S(x, t) = \inf_{y \in \mathbb{R}} \left(\frac{t}{|t|} S_0(y) + \frac{(x-y)^2}{2\,|t|} \right) \tag{26}$$

$$= \inf_{u \in D(u)} \left(\frac{t}{|t|} S_0(x - tu) + \frac{1}{2} |t|\, u^2 \right), \tag{27}$$

or, in a simpler form, for $0 < |t| < \infty$,

$$tS(x, t) = \inf_{y \in \mathbb{R}} \left(tS_0(y) + \frac{(x-y)^2}{2} \right) \tag{28}$$

$$= \inf_{u \in D(u)} \left(tS_0(x - tu) + \frac{1}{2} t^2 u^2 \right), \tag{29}$$

and we have the single-valued relation for all times

$$x(u, t) = tu + u_0^{-1}(u). \tag{30}$$

Notice that, for $t > -\bar{t}_c$, $u(x, t)$ is also single-valued. For $t < -\bar{t}_c$ this is no longer true and we have recourse to the "equal area principle"[6] (Section 3.5, p. 116), which is equivalent to Maxwell's rule in Statistical Mechanics, or to the convex envelope construction of the free energy functional in (25). An example is given in Section 4.

4. APPLICATIONS

We consider the two-parameter family of admissible and compatible initial conditions, with $\alpha \in \mathbb{R}$, $\gamma > 0$,

$$S_0(x) = \alpha x + \frac{1}{\gamma} \ln \cosh(\gamma x) \tag{31}$$

$$v_0(x) = \alpha + \tanh(\gamma x) = \alpha + u_0(x) \tag{32}$$

$$2\rho_0(x) = \gamma \frac{1}{\cosh^2(\gamma x)}. \tag{33}$$

The generalized Hopf–Lax formula becomes

$$tS(x,t)=\inf_{y\in\mathbb{R}}\left(t\left(\alpha y+\frac{1}{\gamma}\ln\cosh(\gamma y)\right)+\frac{(x-y)^2}{2}\right), \tag{34}$$

and, for $t>-\bar{t}_c=-\frac{1}{\gamma}$, y is the unique root of

$$x=y+t(\alpha+\tanh(\gamma y)), \tag{35}$$

or, setting $v=\alpha+u$, $u=\tanh(\gamma y)$,

$$x=t(\alpha+u)+\frac{1}{2\gamma}\ln\frac{1+u}{1-u}. \tag{36}$$

For $t>-\frac{1}{\gamma}$, the smooth solutions of the density are

$$\begin{aligned}2\rho(u,t)=u_x=(x_u)^{-1}&=\left(t+\frac{1}{\gamma}\frac{1}{1-u^2}\right)^{-1}\\&=\frac{\gamma(1-u^2)}{1+\gamma t(1-u^2)},\end{aligned} \tag{37}$$

or else

$$2\rho(x,t)=\gamma(\cosh^2(\gamma(x-\alpha t-tu(x,t)))+\gamma t)^{-1}. \tag{38}$$

At $t=-\bar{t}_c=-\frac{1}{\gamma}$ and in the neighborhood of $u=0$ we have

$$x\simeq-\frac{\alpha}{\gamma}+\frac{1}{3\gamma}u^3, \tag{39}$$

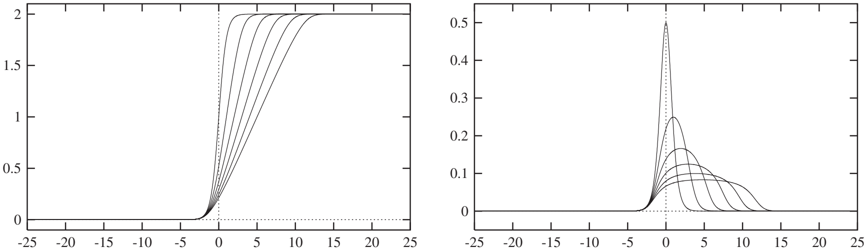
so that

$$u\simeq(3(\gamma x+\alpha))^{\frac{1}{3}}, \tag{40}$$

and thus

$$2\rho=u_x\simeq\frac{\gamma^{\frac{1}{3}}}{(3(x+\frac{\alpha}{\gamma}))^{\frac{2}{3}}}. \tag{41}$$

For $t=-\bar{t}<-\bar{t}_c=-\frac{1}{\gamma}$, (36) still holds and is single-valued for $x(u,t)$ but conversely $u(x,t)$ is not. The density profile consists of a Dirac peak



(a) $v(x)$ at different times. (b) $\rho(x)$ at different times.

Fig. 1. $v(x)$ and $\rho(x)$ for $t = 0, 1, 2, 3, 4, 5$.

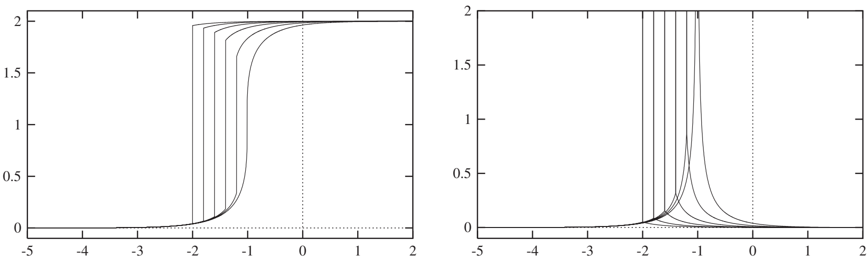
centred at $x = -\alpha\bar{t}$, accompanied by symmetric tails. If $|u_1|$ is the nontrivial solution of $|u_1| = |\tanh(\gamma\bar{t}u_1)|$, i.e., $u_1^2(\gamma\bar{t}) \simeq (\frac{1}{\gamma\bar{t}})^2\,(1-\frac{1}{\gamma\bar{t}})$, then

$$\rho(x,\bar{t}) = |u_1(\gamma\bar{t})|\;\delta(x+\alpha\bar{t}) + \frac{1}{2}\frac{\gamma(1-u^2(x,\bar{t}))}{1-\gamma\bar{t}(1-u^2(x,\bar{t}))}. \tag{42}$$

It is easy to check that $\int_{-\infty}^{\infty}\rho(x,\bar{t})\,\mathrm{d}x = |u_1(\gamma\bar{t})| + (1-|u_1(\gamma\bar{t})|) = 1$.

The figures illustrate the velocity and density profiles for $t = -2$ to $t = +5$, for the case $\alpha = 1$ and $\gamma = 1$.

Figure 1 shows the evolution of v and ρ for times $t = 0, 1, 2, 3, 4, 5 > t_c = -1$. Graphics and are obtained by numerical simulation of the evolution. The initial conditions at $t = 0$ are given by Eqs. (32) and (33), namely $v_0(x) = 1 + \tanh(x)$ and $\rho_0(x) = \frac{1}{2}\frac{1}{\cosh^2(x)}$. The evolution is simulated with an ad-hoc computer program described in ref. 2, Section 3. This program is run with the parameters $N = 8192$, $L = 100$, $\tau = 0.001$.



(a) $v(x)$ at different times. (b) $\rho(x)$ at different times.

Fig. 2. $v(x)$ and $\rho(x)$ for $t = -2, -1.8, -1.6, -1.4, -1.2$ and -1.01.

Figure 2 shows the velocity and density profiles for $t < -1$. Graphics (a) and (b) depict v and ρ respectively for the times $t = -2, -1.8, -1.6, -1.4, -1.2$, and -1.01. The velocity profile is obtained by inverting the relation (36) numerically. For the time interval considered this is done using the fact that for $t \to -\infty$ the velocity reduces to a step-function. The density profile follows explicitly from equation (42).

Figures 1 and 2 give a complete description of the evolution of the model for times $t = -2, -1, \ldots, +5$. Figure 2 gives a refined set of profiles in the interval $[-2; -1]$ using a different scale.

APPENDIX: THE NEWMAN RELATION

Consider a one-dimensional lattice of N equidistant sites occupied for example by Ising spins taking the values ± 1. In the MF picture the ferromagnetic pair interaction $-J_{ij} = -J/N$ and the Hamiltonian is

$$\mathscr{H} = -\frac{1}{2}\frac{J}{N}(M^2 - N) - HM, \tag{43}$$

where M is the total magnetization. With the volume element

$$d\omega = \prod_{i=1}^{N} (\delta(\mu_i - 1) + \delta(\mu_i + 1))\, d\mu_i, \qquad \mu_i \in \mathbb{R}, \tag{44}$$

the partition function is

$$Z(H, J) = \int d\omega \exp\left(\frac{1}{2}\frac{J}{N}(M^2 - N) + HM\right), \tag{45}$$

and

$$m = \frac{1}{N}(\ln Z)_H \equiv \frac{1}{N}\langle M \rangle. \tag{46}$$

We have

$$m_H = \frac{1}{N}(\langle M^2 \rangle - \langle M \rangle^2), \tag{47}$$

$$m_{HH} = \frac{1}{N}(\langle M^3 \rangle - 3\langle M^2 \rangle \langle M \rangle + 2\langle M \rangle^3), \tag{48}$$

and

$$m_J = \frac{1}{2N^2}(\langle M^3\rangle - \langle M\rangle\langle M^2\rangle). \tag{49}$$

Inspection of (46)–(49) gives the Newman relation

$$m_J - m m_H = \frac{1}{2N} m_{HH}. \tag{50}$$

This has to be compared with Burgers' equation for a viscous fluid,

$$u_t + u u_x = \nu u_{xx}, \tag{51}$$

to explain Newman's choice of the mapping $t \leftrightarrow J$ and $x \leftrightarrow -H$ with $\nu \leftrightarrow \frac{1}{2N}$. Thus, the thermodynamic limit in (50) corresponds to the inviscid limit in (51).

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